Conductance Quantization and Landauer Formula

Nina Leonhard
SS 2010
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
Important length scales

- **Wavelength**: at low temperatures current transport occurs for electrons with energies near the Fermi-energy. The Fermi-wavelength is given as:

$$\lambda_F = \frac{2\pi}{k_F}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

For low temperatures, meaning \(kT \ll E_F\), the Fermi distribution is almost a step function.

- GaAs: \(\lambda_F \approx 40\) nm
- Si: \(\lambda_F \approx 35 - 112\) nm
**Wavelength:** at low temperatures current transport occurs for electrons with energies near the Fermi-energy. The Fermi-wavelength is given as:

\[
\lambda_F = \frac{2\pi}{k_F} \quad E_F = \frac{\hbar^2 k_F^2}{2m}
\]

For low temperatures, meaning \( kT \ll E_F \), the Fermi distribution is almost a step function.

GaAs: \( \lambda_F \approx 40 \text{ nm} \)
Si: \( \lambda_F \approx 35 - 112 \text{ nm} \)
Important length scales

- **Mean free path** $L_m$: distance an electron travels until its initial momentum is destroyed
- **Phase-relaxation length** $L_\varphi$: distance an electron travels until its initial phase is randomized

Ballistic regime: $\lambda_F < L < L_m$
Coherent transport: $L < L_\varphi$

GaAs: $L_m \approx 100 - 10000$ nm
Si : $L_m \approx 37 - 118$ nm

Diffusive regime $L > L_m$
Incoherent transport: $L > L_\varphi$

$L_\varphi \approx 200$ nm
$L_\varphi \approx 40 - 400$ nm
Contents

1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
Solve
\[
\left[-\frac{\hbar^2}{2m}\Delta + V(x)\right]\Psi(x) = E\Psi(x)
\]
with boundary conditions
\[
\Psi\left(-\frac{L}{2}\right) = \Psi\left(+\frac{L}{2}\right) = 0
\]
\[
\Rightarrow E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}
\]
\[
k_n = \frac{n\pi}{L}
\]
with \(n \in \mathbb{N}\)

Number of occupied states
\[
E_F = \frac{\hbar^2 k_F^2}{2m} = E_M = \frac{\hbar^2 \pi^2 M^2}{2mL^2}
\]
\[
\Rightarrow M = \text{Int} \left[\frac{k_F L}{\pi}\right]
\]
Potential well

Discrete energy-levels for small widths
For macroscopic systems energy bands
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
Two semiconductor layers:
n-AlGaAs and i-GaAs layer
electrons flow from n- to i-layer
electrons are „caught“ in z-direction
Wavefunction of a free particle

\[ \Psi(x, y, z) = \phi_m(z)e^{ik_xx}e^{ik_yy} \]

Its energy is given as:

\[ E(k) = E_C + \varepsilon_m(z) + \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 \right) \]

\( \varepsilon_m(z) \): transverse energy in z-direction

2-deg: only \( m = 1 \) is occupied

\[ \varepsilon_1(z) < E_F \]

\[ \varepsilon_m>1(z) > E_F \]

\[ k = \sqrt{k_x^2 + k_y^2} \]
2-dimensional electron gas

Ohm’s law for the resistance \( R = \frac{L}{\sigma W} \)

We can also use the conductance \( G = R^{-1} = \frac{\sigma W}{L} \)

For \( L \to 0 \) we would expect \( G \to \infty \)

But: experiments show that \( G \) is quantized

\(|W| \ll L \quad k_F < L < L_m\)
2-dimensional electron gas

First experimental results were obtained by B.J. van Wees in 1988

2-dimensional electron gas at an AlGaAs-GaAs interface
The width is controlled with the gate voltage

![Diagram showing conductance as a function of gate voltage](image)
Because $W$ is so small, only a few modes are occupied. We can rewrite the wavefunction:

$$\Psi(x, y, z) = \phi_1(z) \xi_n(y) e^{ikx}$$

Its energy is given as:

$$E(k) = E_C + \varepsilon_1^{(z)} + \varepsilon_n^{(y)} + \frac{\hbar^2 k^2}{2m}$$

$\varepsilon_n^{(y)}$: transverse energy in $y$-direction (e.g. energies of the potential well)

$$E_C + \varepsilon_1^{(z)} + \varepsilon_n^{(y)} < E_F : \text{open transport channel}$$

$$E_C + \varepsilon_1^{(z)} + \varepsilon_n^{(y)} > E_F : \text{closed transport channel}$$
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
Assumptions:

- **Reflectionless contacts**: the current flowing from the conductor to the contacts is not reflected
- **Ballistic conductor**: no reflection within the conductor
- **Low temperatures**
Result: finite contact resistance that is quantized for a ballistic conductor $G_C = \frac{2e^2}{\hbar} M$

How do we calculate $M$?
Number of modes can be estimated to be (for zero magnetic field)

$$M = \text{Int} \left[ \frac{k_F W}{\pi} \right]$$

because of $E_M = E_F$
Landauer formula

A very large number of modes has to be carried by a few modes.

\[ \text{resistance} = \mu_1 - \mu_2 \]

resistance = contact resistance
Now consider a conductor with two ballistic leads. There is a transmission probability $T$ that an electron crosses the conductor.

$$I_1^+ = \frac{2e}{h} M [\mu_1 - \mu_2] \quad \text{and} \quad I_2^+ = \frac{2e}{h} MT [\mu_1 - \mu_2]$$

$$I_1^- = \frac{2e}{h} M (1 - T) [\mu_1 - \mu_2]$$

Total current: $I = I_1^+ - I_1^- = I_2^+ = \frac{2e}{h} MT [\mu_1 - \mu_2] \Rightarrow G = \frac{2e^2}{h} MT$
Landauer formula

\[ G = \frac{2e^2}{h} MT \]

Generalization: \[ G = \frac{2e^2}{h} \sum_n MT_n \]

Can we obtain Ohm’s law from the Landauer formula? Yes, we can see that:

\[ G^{-1} = \frac{L}{\sigma W} + \frac{L_0}{\sigma W} \]
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
Landauer-Büttiker-formalism

More difficult problem, e.g.
The formula has to be modified

\[ I_p = \frac{2e}{\hbar} \sum_q \left[ \overline{T}_{q\leftarrow p} \mu_p - \overline{T}_{p\leftarrow q} \mu_q \right] \quad \text{with} \quad \overline{T} = MT \]

We can introduce

\[ G_{pq} = \frac{2e^2}{\hbar} \overline{T}_{p\leftarrow q} \]

and obtain

\[ I_p = \sum_q G_{pq} V_p - G_{qp} V_q \]

Sum rule (Kirchhoff-laws):

\[ \sum_q G_{qp} = \sum_q G_{pq} \]
1. Important length scales
2. Potential well
3. 2-dimensional electron gas
4. Landauer formula
5. Landauer-Büttiker formalism
6. S-Matrix
For a coherent conductor the transmission function can be expressed with the scattering matrix.

The scattering matrix relates the incoming amplitudes for each state with the outgoing amplitudes after the scattering process.

$$\{b\} = S\{a\}$$

For the transmission probabilities:

$$T_{m\leftarrow n} = |s_{m\leftarrow n}|^2$$

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$r, t, r', t'$ are matrices
Properties of the S-matrix:

• Calculating the S-matrix is equivalent to solving the problem.

• $\dim S = M_T \times M_T$ with $M_T(E) = \sum_p M_p(E)$

• $S$ has to be unitary $S^\dagger S = SS^\dagger = I$

• Reversing the magnetic field transposes the S-Matrix

$$S(B) = S^t(-B)$$
Combining S-matrices:

Instead of solving the problem rightaway, one can divide it into smaller problems that have already been solved.

Example:
S-Matrix

\[
\begin{align*}
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} r(1) & t'(1) \\ t(1) & r'(1) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_3 \\ a_2 \end{pmatrix} &= \begin{pmatrix} r(2) & t'(2) \\ t(2) & r'(2) \end{pmatrix} \begin{pmatrix} a_3 \\ b_2 \end{pmatrix} \\
\text{Eliminate } a_2 \text{ and } b_2: \quad \begin{pmatrix} b_1 \\ b_3 \end{pmatrix} &= \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} \\
\end{align*}
\]

Where \( t = \frac{t(2)t(1)}{1-r'(1)r(2)} \)

\[
\begin{align*}
t' &= \frac{t'(1)t'(2)}{1-r(2)r'(1)} \\
r &= r(1) + \frac{t'(1)r(2)t(1)}{1-r'(1)r(2)} \\
r' &= r'(2) + \frac{t(2)r'(1)t'(2)}{1-r'(1)r(2)}
\end{align*}
\]
• For mesoscopic ballistic conductors the conductance is \( G_C = \frac{2e^2}{h} M \)

• Conductor with 2 ballistic leads \( G = \frac{2e^2}{h} MT \)

• Generalization for many conductors \( G_{pq} = \frac{2e^2}{h} T_{p\leftarrow q} \)

• We can use the S-matrix to calculate the conductance
Thank you for your attention.